Measuring Systemic Risk: Robust Ranking Techniques Approach

Amirhossein Sadoghi
Practical Quantitative Finance Center
Frankfurt School of Finance - Germany

January 6, 2014
Introduction

1. In recent years, modern financial system has become much more complex, *concentrated* and *interconnected*.

2. Quantifying the complexity and concentration of financial systems is difficult.

3. The **link structure** of this network is an important element to measure Systemic Risk.

4. In our research, we use a **Robust Ranking Technique** to measure systemic risk and reduce the Eigenvalue problem to a convex optimization problem and solve it with an efficient technique.
What is systemic risk?

Systemic risk can be defined as the risk of default of a large fraction of the financial system regarding to interlinked financial exposures among institutions.

1. One of the main sources of systemic risk is the *contagion* of economic distress in the financial system.

2. This financial distress can spread throughout the whole of financial system in a *domino fashion*.

3. Network approach well suited for dealing with *heterogeneity* of assumptions, charting the *dynamic propagation* of shocks within the financial system.

4. Effect of connectivity of financial network as *risk absorbers or amplifiers*
The financial system can be modeled as a weighted, directed graph whose nodes are financial institutions.

1. (Directed) links represent counterparty exposures: \( A_{ij} \) is the exposure of \( i \) to \( j \)

2. Capital \( Cpa_i \) Tier I capital, proxy for absorbing market losses.
Definition: (Susceptibility Weight)

The **Susceptibility Weight** of a node is the fraction of capital wiped out by the default of a single counterparty.

\[ w_i = \frac{A_{ij}}{Cp_ia_i} \]  

A node may become insolvent due to the default of a single counterparty.

**Solvency Index**

The default of \( i \) the following cash flows over a short term horizon because of debts are collected from its debtors

\[ Solv_i = \sum_i A_{ij} - \lambda \sum_j A_{ij} \]  

\( \lambda \) Proportion of deposits withdrawn by hoarding banks
Problem Setup

Relative Impact Index

The economic value of the impact of \( i \) on \( j \) by multiplying the impact by the relative economic value of the node \( j \)

\[
    r_i = \frac{Solv_i}{\sum_j Solv_i} \\
    j \in s \mid A_{is} > 0
\]  

(3)

(4)

Contagion Index

Similar to [Battiston 2012] Contagion Index is a measure of the expected loss generated by the failure of the set of institutions.

\[
    I_i = \alpha \sum_j w_{ij} I_j + \beta \sum_j w_{ij} r_i
\]  

(5)
Problem Formulation

In matrix notation:

\[ I = \alpha W \cdot I + \beta W \cdot r \]  \hspace{1cm} (6)

Let’s define matrix \( M \in \mathbb{R}^{n \times n} \) is non-negative (positive) as follows:

\[ M = \alpha W + \beta W \cdot r \]  \hspace{1cm} (7)

\[ I = M \cdot I \]  \hspace{1cm} (8)

They are several issues related using standard Eigenvector centrality and PageRank methods to apply defined problem:

1. Matrix \( M \) is neither column-stochastic nor row-stochastic
2. Vector \( I \) is not unique
3. Financial networks are large weighted networks
We are looking for $\bar{I}$ as a robust extension of the dominant Eigenvector:

$$\bar{I} = M \cdot \bar{I}$$ \hspace{1cm} (9)

We say that the vector $\bar{I}$ is a robust solution of the Eigenvector problem on $M$ if

$$\bar{I} \in \text{Argmin}_{I \in \sum} (\max_{M \subset P} \| \bar{I} - M \cdot \bar{I} \|)$$ \hspace{1cm} (10)

where $\sum = \{ u^2 \in R^{n \times n} | \sum_i u_i = 1 \}$

2. Large-scale problems: mirror-descent family

[A. Juditsky and B. Polyak, 2012] find numerical approximations to the eigenvalues problem
Solving optimization problem

Proposition

Contagion Index for each node of a (directed) interbank networks for each iteration of solution procedure can be determined as:

\[ I_{k+1} = \frac{k}{k+1} WI_k + \frac{1}{k+1} Wr_1 \]

We apply the power method with averaging to minimize

\[ I_k = \frac{I_1 + M \cdot I_1 + \cdots + M^{k-1} \cdot I_1}{k} \]

\[ \| \bar{I} - M \cdot \bar{I} \|_2 = \left\| \frac{M^K I_1 - I_1}{k} \right\|_2 \leq \frac{\text{Constant}}{k} \]

With \( M = \alpha \cdot W + \beta W \cdot r \) and with choosing \( I_1 = W \cdot r_1 \)

\[ I_{k+1} = \frac{k}{k+1} WI_k + \frac{1}{k+1} Wr_1 \]
Simulation model of financial systems

Algorithm 1

1. begin: $I_1$
   
   $I_1 = W \cdot r_1$

2. $k$-th iteration: $I_k = M \cdot I_{k+1}$
   
   where
   
   $M = \frac{k}{k+1} W + \frac{1}{k+1} W \cdot r$

3. Stop condition: $|I_k - \bar{I}| < \text{Tolerance}$

Simulation Steps

1. Generate BA scale-free directed networks from properties of empirical study of Brazilian networks [Cont, R., Moussa, 2010].

2. Generate Complete Networks.

3. Heterogeneity: Use Pareto distribution for connectivities and exposure sizes.
### Numerical Results

**Table:** Numerical results of BA Network

<table>
<thead>
<tr>
<th>BA Network</th>
<th>Size</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interco</td>
<td>4821</td>
<td>10475</td>
<td></td>
</tr>
<tr>
<td>Max in</td>
<td>222</td>
<td>403</td>
<td></td>
</tr>
<tr>
<td>Max out</td>
<td>214</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>Tol</td>
<td>$10^{-3}$</td>
<td>$10^{-4}$</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>time</td>
<td>0.0039</td>
<td>0.0075</td>
<td>0.026</td>
</tr>
<tr>
<td>Ite</td>
<td>24</td>
<td>77</td>
<td>244</td>
</tr>
<tr>
<td>Rel-Err</td>
<td>0.054</td>
<td>0.053</td>
<td>0.050</td>
</tr>
</tbody>
</table>

\[
Rel - Err = \frac{(Err_{Alg1} - Err_{interior})}{Err_{interior}}
\]
### Numerical Results

**Table: Numerical results of Complete Network**

<table>
<thead>
<tr>
<th></th>
<th>Complete</th>
<th>Network</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Size</strong></td>
<td>100</td>
<td>300</td>
</tr>
<tr>
<td><strong>Interco</strong></td>
<td>39474</td>
<td>359039</td>
</tr>
<tr>
<td><strong>Max in</strong></td>
<td>443</td>
<td>1311</td>
</tr>
<tr>
<td><strong>Max out</strong></td>
<td>453</td>
<td>1290</td>
</tr>
<tr>
<td><strong>Tol</strong></td>
<td>$10^{-3}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td><strong>time</strong></td>
<td>0.002</td>
<td>0.005</td>
</tr>
<tr>
<td><strong>Ite</strong></td>
<td>16</td>
<td>51</td>
</tr>
<tr>
<td><strong>Rel-Err</strong></td>
<td>0.050</td>
<td>0.049</td>
</tr>
</tbody>
</table>

\[
Rel - Err = \frac{Err_{Alg1} - Err_{interior}}{Err_{interior}}
\]
## Numerical Results

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>log(Impact Index)</td>
<td>log(Impact Index)</td>
<td>log(Impact Index)</td>
</tr>
<tr>
<td>log(Size node)</td>
<td>-0.350</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.319)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(Total In connection)</td>
<td></td>
<td>1.369***</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.130)</td>
<td></td>
</tr>
<tr>
<td>log(Total Out connection)</td>
<td></td>
<td></td>
<td>1.334***</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.104)</td>
</tr>
<tr>
<td>Constant</td>
<td>-10.37***</td>
<td>-15.89***</td>
<td>-15.71***</td>
</tr>
<tr>
<td></td>
<td>(1.255)</td>
<td>(0.417)</td>
<td>(0.331)</td>
</tr>
<tr>
<td>Observations</td>
<td>74</td>
<td>74</td>
<td>74</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.003</td>
<td>0.599</td>
<td>0.691</td>
</tr>
</tbody>
</table>

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$
Numerical Results
We define a metric for **systemic importance of a financial institution** in financial system.

This metric can be used to model the contagion of **illiquidity** and **insolvency**

Nodes with high systemic impact seem to be those who are **major counterparties** are large and **well-connected** nodes.

We need efficient methods for **large scale simulation** and identify importance of a financial institution in these networks and helping to better understand of systemic risk
References

4. B.T. Polyak, A.V. Timonina, PageRank: new regularizations and simulation models, 18th IFAC World Congress, 11202–11207, Milan, Italy, August 2011
References 2


Thank you for your attention
Robust Dominant Eigenvector

Perron-Frobenius Theorem

states that there exist a robust dominant eigenvector of matrix $M$ which is belong to the set of non-negative (positive) matrix $P$

and $\|\ldots\|$ is some norms

$$\bar{I} \in \text{Argmin}_{I \in \Sigma} \left( \|\bar{I} - M \cdot \bar{I}\|_2 + \epsilon \right) \quad (14)$$

$\bar{I}$ coincides with $I$ if $\epsilon$ is small enough and all eigenvalues lay inside the $1 - \lambda_i > 0; i = 2; \ldots; n$

Proof:[A. Juditsky and B. Polyak,2012, Proposition 2]

In the case norm 2 vector $\bar{I}$ is unique due to the strict convexity on $R^{n \times n}$
Solving optimization problem

The eigenvalue problem is reduced to a convex optimization problem.

**The power method**

The power method can be used when:
1. has linearly independent eigenvectors
2. The eigenvalues can be ordered in magnitude a

\[
|λ_1| > |λ_2| > \cdots > |λ_n| \tag{15}
\]

\[\sigma_{k+1} = \text{dominate term in } M \cdot I_k \]

\[I_{k+1} = \left(1/σ_{k+1}\right)M \cdot I_k\]

If \(I_0\) is chosen appropriately,
Financial Networks Example

Nodes are banks and links between banks their financial flows. Link widths scale with value of payments and node sizes scale with the capital value.
Example
Basic BA Algorithm

The Barabasi-Albert (BA) algorithm for generating random scale-free networks [R. Albert; A.-L. Barabasi ,2002]:

BA Algorithm

1. Start with an initial set of small fully connected nodes
2. Add new vertices one by one, each one with exactly m edges
3. Each new edge connects to an existing vertex in proportion to the number of edges that vertex already has preferential attachment