Asymptotics of forward implied volatility

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Based on joint works with Patrick Roome (Imperial College London):

- Large-maturity regimes of the Heston forward smile. *In progress.*
(Spot) implied volatility

- Asset price process: \((S_t = e^{X_t})_{t \geq 0}\), with \(X_0 = 0\).
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:
  \[
  C_{\text{BS}}(\tau, k, \sigma) := \mathbb{E}_0 \left( e^{X_\tau} - e^k \right)_+ = \mathcal{N}(d_+) - e^k \mathcal{N}(d_-),
  \]
  where
  \[
  d_{\pm} := -\frac{k}{\sigma \sqrt{\tau}} \pm \frac{1}{2} \sigma \sqrt{\tau}.
  \]
- Spot implied volatility \(\sigma_{\tau}(k)\): the unique (non-negative) solution to
  \[
  C_{\text{observed}}(\tau, k) = C_{\text{BS}}(\tau, k, \sigma_{\tau}(k)).
  \]
- Spot implied volatility: unit-free measure of option prices.
- However not available in closed form for most models.

Antoine Jacquier

Asymptotics of forward implied volatility
Spot implied volatility ($\sigma_T(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Forde et al. (2012), Jacquier et al. (2012): small- and large-$\tau$ using large deviations and saddlepoint methods.
- Mijatović-Tankov (2012): small-$\tau$ for jump models.

Related works:

- Kim, Kunitomo, Osajima, Takahashi (1999-...): asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).

Note: from expansions of densities to implied volatility asymptotics is ‘automatic’ (Gao-Lee (2013)).
Forward implied volatility

- Fix $t > 0$: forward-starting date; $\tau > 0$: remaining maturity.
- Forward-start call option is a European call option with payoff
  \[
  \left( \frac{S_{t+\tau}}{S_t} - e^k \right)^+ = \left( e^{X_{t+\tau}} - x_t - e^k \right)^+,
  \]
  and value today
  \[
  \mathbb{E}_0 \left( e^{X_{t+\tau}} - x_t - e^k \right)^+.
  \]
- BSM model: its value today is simply worth $C_{BS}(\tau, k, \sigma)$ (stationary increments).
- Forward implied volatility $\sigma_{t, \tau}(k)$: the unique solution to
  \[
  C_{\text{observed}}(t, \tau, k) = C_{BS}(\tau, k, \sigma_{t, \tau}(k)).
  \]
- Obviously, $\sigma_{0, \tau}(k) = \sigma_{\tau}(k)$.
- Alternative definition: $\left( S_{t+\tau} - S_t e^k \right)^+$. 
Introduction

General results

Applications

Heston small-maturity

Mathematical problem

Motivation

Calibration:

- Forward-start options serve as natural hedging instruments for many exotic securities and it is therefore important for a model to be able to calibrate to liquid forward smiles.

Model Risk:

- Calibrate two different models to some observed spot implied volatility smiles: perfect calibration. Use these calibrated models to price some ‘exotic’ options, say barrier options: two different prices. One of the reasons: subtle dependence on the dynamics of implied volatility smiles.

- One metric that can be used to understand the dynamics of implied volatility smiles (Bergomi(2004) calls it a ‘global measure’ of the dynamics of implied volatilities) is to use the forward smile defined above.
Existing literature on forward smiles

- Glasserman and Wu (2011): different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility.
- Keller-Ressel (2011): when the forward-start date $t$ becomes large ($\tau$ fixed).
- Bompis-Hok (2013): expansion in local volatility models.
Today's menu

- Asymptotics in time ($t$) / maturity ($\tau$) of forward implied volatility smiles.
- What will not be covered: connections with VIX / variance swaps.
Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

\[
\begin{align*}
    dX_t &= -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t, \quad X_0 = 0, \\
    d\sigma_t &= (a + b\sigma_t) dt + \xi dZ_t, \quad \sigma_0 = \sigma_0 > 0,
\end{align*}
\]
Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

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    d\sigma_t &= (a + b\sigma_t) dt + \xi dZ_t, \quad \sigma_0 = \sigma_0 > 0,
\end{align*} \]

Define \( X_{\tau}^{(t)} := X_{t+\tau} - X_t \), then

\[ \begin{align*}
    dX_{\tau}^{(t)} &= -\frac{1}{2} (\sigma_{\tau}^{(t)})^2 d\tau + \sigma_{\tau}^{(t)} dW_{\tau}, \quad X_{0}^{(t)} = 0, \\
    d\sigma_{\tau}^{(t)} &= (a + b\sigma_{\tau}^{(t)}) d\tau + \xi dZ_{\tau}, \quad \sigma_{0}^{(t)} \sim \sigma_t,
\end{align*} \]
Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

\[
\begin{align*}
\text{d}X_t &= -\frac{1}{2} \sigma_t^2 \text{d}t + \sigma_t \text{d}W_t, \quad X_0 = 0, \\
\text{d}\sigma_t &= (a + b\sigma_t) \text{d}t + \xi \text{d}Z_t, \quad \sigma_0 = \sigma_0 > 0,
\end{align*}
\]

Define \(X^{(t)}_\tau := X_{t+\tau} - X_t\), then

\[
\begin{align*}
\text{d}X^{(t)}_\tau &= -\frac{1}{2} (\sigma^{(t)}_\tau)^2 \text{d}\tau + \sigma^{(t)}_\tau \text{d}W_\tau, \quad X^{(t)}_0 = 0, \\
\text{d}\sigma^{(t)}_\tau &= (a + b\sigma^{(t)}_\tau) \text{d}\tau + \xi \text{d}Z_\tau, \quad \sigma^{(t)}_0 \sim \sigma_t,
\end{align*}
\]

**Pricing Fwd-start options:**

\[
\mathbb{E}_0(e^{X^{(t)}_\tau} - e^k)^+ = \mathbb{E}_0 \left\{ \mathbb{E}_t(e^{X^{(t)}_\tau} - e^k|\sigma_t)^+ \right\}
\]

**Problem:** Known expansions (as \(\tau \downarrow 0\)) are NOT uniform in space.

*Easy case:* \(\sigma_t\) has compact support, e.g. finite-state Markov chain.
General results: framework

\( (Y_\varepsilon) \): (general) stochastic process. Denote the re-normalised log moment generating function (lmgf) by \( \Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{uY_\varepsilon}{\varepsilon} \right) \right] \), for all \( u \in \mathcal{D}_\varepsilon \subseteq \mathbb{R} \).

We require the following assumptions (Assumption OA) on the behaviour of \( \Lambda_\varepsilon \):

(i) **Expansion property:** \( \Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3) \) holds for \( u \in \mathcal{D}_0^0 \) as \( \varepsilon \downarrow 0 \);

(ii) **Differentiability:** The map \((\varepsilon, u) \mapsto \Lambda_\varepsilon(u)\) is of class \( C^\infty \) on \((0, \varepsilon_0) \times \mathcal{D}_0^0 \);

(iii) **Non-degenerate interior:** \( 0 \in \mathcal{D}_0^0 \);

(iv) **Essential smoothness:** \( \Lambda_0 \) is strictly convex and essentially smooth on \( \mathcal{D}_0^0 \);

(v) **Tail error control:** For any fixed \( p_r \in \mathcal{D}_0^0 \backslash \{0\} \),

(a) \( \Re (\Lambda_\varepsilon (ip_i + p_r)) = \Re (\Lambda_0 (ip_i + p_r)) + \mathcal{O}(\varepsilon) \), for any \( p_i \in \mathbb{R} \);

(b) \( L : \mathbb{R} \ni p_i \mapsto \Re (\Lambda_0 (ip_i + p_r)) \) has a unique maximum at zero and is bounded away from \( L(0) \) as \( |p_i| \) tends to infinity;

(c) \( \Re [\Lambda_\varepsilon(ip_i + p_r) - \Lambda_0(ip_i + p_r)] \leq M \varepsilon \), for some \( M > 0 \), for large \(|p_i|\) and small \( \varepsilon \).

**Note:** (i)-(iv) are Gärtner-Ellis assumptions for large deviations:

\[
\mathbb{P}(Y_\varepsilon \in A) \sim \exp \left( -\frac{1}{\varepsilon} \inf \{ \Lambda^*(x) : x \in A \} \right),
\]

for \( A \subset \mathbb{R} \), as \( \varepsilon \downarrow 0 \), \( \Lambda^* \): dual of \( \Lambda_0 \).
Main result: Option price asymptotics

Theorem (J-Roome, 2013)

Let \(( Y_\varepsilon )\) satisfy OA and \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function satisfying \( f(\varepsilon)\varepsilon = c + O(\varepsilon) \), for some constant \( c \geq 0 \) as \( \varepsilon \downarrow 0 \). For \( k > \Lambda'_0(c) \), as \( \varepsilon \downarrow 0 \),

\[
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{kf(\varepsilon)} \right)^+ = \psi(k, c, \varepsilon) \exp \left\{ -\frac{\Lambda^*(k)}{\varepsilon} + kf(\varepsilon) \right\} \left[ 1 + \alpha_1(k, c)\varepsilon + O(\varepsilon^2) \right],
\]

where \( \psi(k, c, \varepsilon) \equiv \alpha(k, c) \left( c\sqrt{\varepsilon}\mathbf{1}_{\{c>0\}} + \varepsilon^{3/2}f(\varepsilon)\mathbf{1}_{\{c=0\}} \right) \), and \( \Lambda^* : \mathbb{R} \to \mathbb{R}_+ \) is the Fenchel-Legendre transform of \( \Lambda_0 \):

\[
\Lambda^*(k) := \sup_{u \in \mathcal{D}_0} \left\{ uk - \Lambda_0(u) \right\}, \quad \text{for all } k \in \mathbb{R}.
\]

We shall denote \( u^*(k) \) the corresponding saddlepoint: \( \Lambda^*(k) = u^*(k)k - \Lambda_0(u^*(k)) \).

Analogous results hold for Put options when \( k < \Lambda'_0(c) \) and for covered calls when \( k \in (\Lambda'_0(0), \Lambda'_0(c)) \).
Application I: forward-start large-maturity

Recall the fwd-start process: \(X^{(t)}_{\tau} := X_{t+\tau} - X_t\). Let \((Y_\varepsilon) := (\varepsilon X^{(t)}_{\tau/\varepsilon})\) and \(f(\varepsilon) \equiv 1/\varepsilon:\)

\[
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{k f(\varepsilon)} \right)^+ = \mathbb{E} \left( e^{X^{(t)}_{\tau/\varepsilon}} - e^{k/\varepsilon} \right)^+
\]

**Corollary (Large-maturity, \(t \geq 0\))**

If \((\tau^{-1} X^{(t)}_{\tau})_{\tau > 0}\) satisfies OA with \(\varepsilon = \tau^{-1}\) and \(1 \in D_0^0\), then for \(k > \Lambda'_0(1)\), as \(\tau \uparrow \infty\):

\[
\mathbb{E}_0 \left( e^{X^{(t)}_{\tau}} - e^{k\tau} \right)^+ = \frac{e^{-\tau(\Lambda^*(k) - k) + \Lambda_1(u^*(k)) \tau^{-1/2}}}{u^*(k)(u^*(k) - 1) \sqrt{2\pi \Lambda'_0(u^*(k))}} \left[ 1 + \frac{\alpha_1(k)}{\tau} + O \left( \frac{1}{\tau^2} \right) \right]
\]

**Corollary:** \((\tau^{-1} X^{(t)}_{\tau})_{\tau}\) satisfies a LDP with speed \(\tau^{-1}\) and rate function

\(k \mapsto \Lambda^*(k) - k\).

When \(t = 0\), we recover Jacquier, Keller-Ressel, Mijatović (2013), which intuitively makes sense.
Application II: forward-start diagonal small-maturity

Recall the fwd-start process: $X_T^{(t)} := X_{t+T} - X_t$. Let $(Y_\varepsilon) := (X_{\varepsilon T}^{(\varepsilon t)})$ and $f(\varepsilon) \equiv 1$:

$$
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{k f(\varepsilon)} \right)^+ = \mathbb{E} \left( e^{X_{\varepsilon T}^{(\varepsilon t)}} - e^{k} \right)^+
$$

**Corollary (Diagonal small-maturity, $t, \tau > 0$)**

If $(X_{\varepsilon T}^{(\varepsilon t)})_{\varepsilon > 0}$ satisfies OA, then the following holds for $k > \Lambda'_0(0)$, as $\varepsilon \downarrow 0$:

$$
\mathbb{E}_0 \left( e^{X_{\varepsilon T}^{(\varepsilon t)}} - e^{k} \right)^+ = \frac{\exp \left\{ -\frac{\Lambda^*(k)}{\varepsilon} + k + \Lambda_1(u^*(k)) \right\} \varepsilon^{3/2}}{u^*(k)^2 \sqrt{2\pi \Lambda''_0(u^*(k))}} \left[ 1 + \alpha_1(k) \varepsilon + O(\varepsilon^2) \right].
$$

**Corollary**: $(X_{\varepsilon T}^{(\varepsilon t)})_{\varepsilon > 0}$ satisfies a LDP with speed $\varepsilon$ and rate function $\Lambda^*$. 
Corollary I: large-maturity forward smile

**Corollary (Large-maturity forward smile asymptotics)**

If \((\tau^{-1}X_{\tau}^{(t)})_{\tau > 0}\) satisfies OA with \(\varepsilon = \tau^{-1}\) and \(\Lambda_0(1) = 0\) with \(1 \in \mathcal{D}_0^o\), then for all \(k \in \mathbb{R}\), as \(\tau \uparrow \infty\):

\[
\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k, t) + \frac{v_1^\infty(k, t)}{\tau} + \frac{v_2^\infty(k, t)}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right),
\]

where \(v_0^\infty(\cdot, t)\), \(v_1^\infty(\cdot, t)\) and \(v_2^\infty(\cdot, t)\) are continuous functions on \(\mathbb{R}\).

- If \(S = e^X\) is a true martingale, then \(\Lambda_0(1) = 0\).
- For \(t = 0\) (spot smiles), we recover Jacquier, Keller-Ressel, Mijatović (2013) in the case of affine stochastic volatility models with jumps.
Corollary II: diagonal small-maturity forward smile

Corollary (Diagonal small-maturity forward smile asymptotics)

If \((X^{(\varepsilon \tau)}_{\varepsilon})_{\varepsilon > 0}\) satisfies OA and \(\Lambda'_0(0) = 0\), then for all \(k \in \mathbb{R}\), as \(\varepsilon \downarrow 0\),

\[
\sigma^2_{\varepsilon t, \varepsilon \tau}(k) = v_0(k, t, \tau) + v_1(k, t, \tau)\varepsilon + v_2(k, t, \tau)\varepsilon^2 + O(\varepsilon^3),
\]

where \(v_0(\cdot, t, \tau), v_1(\cdot, t, \tau)\) and \(v_2(\cdot, t, \tau)\) are continuous functions on \(\mathbb{R}\).

Under the assumption \(\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + O(\varepsilon^3)\), as \(\varepsilon \downarrow 0\), \(v_0, v_1,\) and \(v_2\) depend on the derivatives of \(\Lambda_0, \Lambda_1\) and \(\Lambda_2\) evaluated at \(u^*(k)\).
When \(t = 0\) (spot smiles), we recover Forde-Jacquier-Lee (2012), Gao-Lee (2013), Berestycki-Busca-Florent (2004).
Examples

- \( S \): exponential Lévy model. Stationary increment property implies \( \sigma_{t,\tau} \) does not depend on \( t \).
- \( S \): time-changed exponential Lévy model.
- Stochastic volatility models; Schöbel-Zhu: \( d\sqrt{V_t} = \kappa(\theta - \sqrt{V_t})dt + \xi dZ_t \).
- Heston (affine stochastic volatility) model:

\[
\begin{align*}
    dX_t &= -\frac{1}{2} V_t dt + \sqrt{V_t} dW_t, \quad X_0 = 0, \\
    dV_t &= \kappa(\theta - V_t) dt + \xi \sqrt{V_t} dZ_t, \quad V_0 = \nu > 0, \\
    d\langle W, Z \rangle_t &= \rho dt,
\end{align*}
\]

with \( \kappa > 0, \xi > 0, \theta > 0 \) and \(|\rho| < 1\).
Heston diagonal small-maturity

- We can compare spot and forward (diagonal) small-maturity smiles:

\[
\sigma_{\varepsilon t, \varepsilon \tau}(0) = \sigma_{0, \varepsilon \tau}(0) - \frac{\varepsilon t}{8 \sqrt{\nu}} (\xi^2 + 4\kappa (\nu - \theta)) + O(\varepsilon^2),
\]

\[
\partial_k^2 \sigma_{\varepsilon t, \varepsilon \tau}(0) = \partial_k^2 \sigma_{0, \varepsilon \tau}(0) + \frac{\xi^2 t}{4 \tau \nu^{3/2}} + O(\varepsilon).
\]

- At zeroth order in \( \varepsilon \) the wings of the forward smile increase to arbitrarily high levels with decreasing maturity.

- Bühler (2002): ‘Heston implied forward volatility: short skew becomes U-shaped, which is inconsistent with observations.’
Figure: In (a) circles, squares and diamonds represent the zeroth, first- and second-order asymptotics respectively and triangles represent the true forward smile using Fourier inversion. In (b): differences between the true forward smile and the asymptotic. We use $t = 1/2$ and $	au = 1/12$ and the Heston parameters $v = 0.07$, $\theta = 0.07$, $\kappa = 1$, $\xi = 0.34$, $\rho = -0.8$.

Copyright: Numerics performed with the IPython-based Zanadu interface provided by Zeliade Systems.
Heston small-maturity asymptotics: problem overview

Consider now fixed $t > 0$, and $\tau \downarrow 0$. The framework above does not apply.
Heston small-maturity asymptotics: problem overview

Consider now fixed $t > 0$, and $\tau \downarrow 0$. The framework above does not apply. Rescaled log mgf: $\Lambda^{(t)}(t, a) := a \log \mathbb{E} \left( e^{uX^{(t)}_\tau} / a \right)$.

**Lemma**

If $h(\tau) \equiv a \sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda^{(t)}(t, h(\tau)) = 0$, for $|u| < a / \sqrt{\beta_t}$ and $\infty$ otherwise.

Define $\Lambda(u) := \lim_{\tau \downarrow 0} \Lambda^{(t)}(t, \tau)$ on $\mathcal{D}_\Lambda$, and its Fenchel-Legendre transform $\Lambda^*(k) := \sup \{ uk - \Lambda(u), u \in \mathcal{D}_\Lambda \}$.

**Corollary**

$\Lambda^*(k) = |k| / \sqrt{\beta_t}$, for all $k \in \mathbb{R}$.

Clearly, no convexity argument holds here.
Main result

Theorem (J-Roome, 2012)

Let $t > 0$. In the Heston model, the following expansion holds for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$: 

$$
\mathbb{E} \left( e^{X^{(t)}_{\tau}} - e^k \right)^+ = (1 - e^k) \mathbb{1}_{\{k < 0\}} 
+ \exp \left( \frac{-\Lambda^*(k)}{\sqrt{\tau}} + \frac{c_0(k)}{\tau^{1/4}} + c_1(k) + k \right) \tau^{7/8} \frac{-\kappa \theta}{2\xi^2} c_2(k) \left[ 1 + c_3(k) \tau^{1/4} + o(\tau^{1/4}) \right].
$$

Corollary: $(X^{(t)}_{\tau})_{\tau \geq 0}$ satisfies a LDP with speed $\sqrt{\tau}$ and rate function $\Lambda^*$ as $\tau \downarrow 0$. Compare with (see Forde-Jacquier-Lee (2012)), when $t = 0$: 

• 

$$
\mathbb{E} \left( e^{X^{(0)}_{\tau}} - e^k \right)^+ = (1 - e^k) \mathbb{1}_{\{k < 0\}} + \exp \left( \frac{\Lambda^*(k)}{\tau} \right) \tau^{3/2} c_2(k) (1 + O(\tau)),
$$

• $(X^{(0)}_{\tau})_{\tau \geq 0}$ satisfies a LDP with speed $\tau$ and good rate function $\Lambda^*$ as $\tau \downarrow 0$. 
Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$\sigma_{t,\tau}^2(k) = \begin{cases} 
\displaystyle \frac{\nu_0(k, t)}{\tau^{1/2}} + \frac{\nu_1(k, t)}{\tau^{1/4}} + o \left( \frac{1}{\tau^{1/4}} \right), & \text{if } 4\kappa\theta \neq \xi^2, \\
\displaystyle \frac{\nu_0(k, t)}{\tau^{1/2}} + \frac{\nu_1(k, t)}{\tau^{1/4}} + \nu_2(k, t) + \nu_3(k, t)\tau^{1/4} + o \left( \tau^{1/4} \right), & \text{if } 4\kappa\theta = \xi^2.
\end{cases}$$

Compare with the $t = 0$ case: $\sigma_{0,\tau}^2(k) = \sigma_0^2(k) + a(k)\tau + o(\tau)$, when $k \neq 0$. 
Small-maturity smile

Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$
\sigma_{t,\tau}^2(k) = \begin{cases} 
\frac{v_0(k, t)}{\tau^{1/2}} + \frac{v_1(k, t)}{\tau^{1/4}} + o\left(\frac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\
\frac{v_0(k, t)}{\tau^{1/2}} + \frac{v_1(k, t)}{\tau^{1/4}} + v_2(k, t) + v_3(k, t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2.
\end{cases}
$$

Compare with the $t = 0$ case: $\sigma_{0,\tau}^2(k) = \sigma_0^2(k) + a(k)\tau + o(\tau)$, when $k \neq 0$.

At-the-money case $k = 0$, $t > 0$.

As $\tau \downarrow 0$,

$$
\sigma_{t,\tau}(0) = \begin{cases} 
\Delta_0(t) + o(1), & \text{if } 4\kappa\theta \leq \xi^2, \\
\Delta_0(t) + \Delta_1(t)\tau + o(\tau), & \text{if } 4\kappa\theta > \xi^2.
\end{cases}
$$
**Numerics**

**Figure:** Here $t = 1$ and $\tau = 1/12$. In (a) circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively and backwards triangles represent the true forward smile using Fourier inversion. In (b) we plot the errors.
Sketch of proof: large deviations analysis
Step 1: Find the speed of convergence

Rescaled log mgf: $\Lambda^{(t)}(u, a) := a \log \mathbb{E} \left( e^{uX^{(t)}_\tau} / a \right)$.

**Lemma**

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ continuous with $\lim_{\tau \downarrow 0} h(\tau) = 0$ and $a > 0$. Then

(i) If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda^{(t)}(u, h(\tau)) = 0$, for $|u| < a/\sqrt{\beta_t}$ and $\infty$ otherwise;

(ii) if $\sqrt{\tau}/h(\tau) \uparrow \infty$ then $\lim_{\tau \downarrow 0} \Lambda^{(t)}(u, h(\tau)) = 0$, for $u = 0$ and $\infty$ otherwise;

(iii) if $\sqrt{\tau}/h(\tau) \downarrow 0$ then $\lim_{\tau \downarrow 0} \Lambda^{(t)}(u, h(\tau)) = 0$, for all $u \in \mathbb{R}$.

(i) is the only non-trivial zero limit and $D_\Lambda = (-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t})$.

Define $\Lambda(u) := \lim_{\tau \downarrow 0} \Lambda^{(t)}(u)$ on $D_\Lambda$, and its Fenchel-Legendre transform $\Lambda^*(k) := \sup\{uk - \Lambda(u), u \in D_\Lambda\}$.

**Lemma**

$\Lambda^*(k) = |k|/\sqrt{\beta_t}$, for all $k \in \mathbb{R}$.
Step 2: Weak convergence of (rescaled) measure

Consider the saddlepoint equation (*): \( \frac{\partial}{\partial u} \Lambda_{\tau}^{(t)}(u_{\tau}^*(k)) = k \).

**Lemma**
For any \( k \neq 0, \tau > 0 \), (*) admits a unique solution \( u_{\tau}^*(k) \), and
\[
 u_{\tau}^*(k) = a_0(k) + a_1(k)\tau^{1/4} + a_2(k)\tau^{1/2} + a_3(k)\tau^{3/4} + O(\tau) \in \mathcal{D}_\Lambda^0, \text{ as } \tau \downarrow 0.
\]
For small \( \tau \), introduce the time-dependent change of measure
\[
 \frac{dQ_{k, \tau}}{dP} \coloneqq \exp \left( \frac{u_{\tau}^*(k)X_{\tau}^{(t)}}{\sqrt{\tau}} - \frac{\Lambda_{\tau}^{(t)}(u_{\tau}^*(k))}{\sqrt{\tau}} \right).
\]

Define \( Z_{\tau, k} := (X_{\tau}^{(t)} - k)/\tau^{1/8} \) and \( \Phi_{\tau, k}(u) \coloneqq \mathbb{E}_{Q_{k, \tau}} \left( e^{iuZ_{\tau, k}} \right) \).

**Lemma**
The following expansion holds for all \( k \neq 0 \) as \( \tau \downarrow 0 \):
\[
 \Phi_{\tau, k}(u) = e^{-\frac{1}{2} \zeta^2(k)u^2} \left[ 1 + \phi_1(k, u)\tau^{1/8} + \phi_2(k, u)\tau^{1/4} + O(\tau^{3/8}) \right]. \quad (1)
\]

**Corollary**: \( Z_{\tau, k} \) converges weakly to \( \mathcal{N}(0, \zeta(k)^2) \) under \( Q_{k, \tau} \).
Step 3: Wrapping up

\[ E \left[ e^{X^T} - e^k \right]^+ = \mathbb{E}_{Q, \tau}^{k, \tau} \left[ \frac{dQ_{k, \tau}}{dP} \left\{ e^{X^T} - e^k \right\}^+ \right] = e^{\frac{\Lambda^T(u^*)}{\sqrt{\tau}}} \mathbb{E}_{Q, \tau}^{k, \tau} \left[ e^{-\frac{u^* X^T}{\sqrt{\tau}}} \left\{ e^{X^T} - e^k \right\}^+ \right] = e^{-\frac{ku* - \Lambda^T(u^*)}{\sqrt{\tau}}} e^k \mathbb{E}_{Q, \tau}^{k, \tau} \left[ e^{-\frac{u^* Z^T, k}{\tau^{3/8}}} \left( e^{Z^T, k\tau^{1/8}} - 1 \right)^+ \right]. \]

Final steps, take the Fourier transform:

\[ \mathcal{F} \left( e^{-\frac{u^* Z^T, k}{\tau^{3/8}}} \left( e^{Z^T, k\tau^{1/8}} - 1 \right)^+ \right)(u) = C_{k, \tau}(u) \]

use Parseval’s identity (or so):

\[ \mathbb{E} \left[ e^{-\frac{u^* Z^T, k}{\tau^{3/8}}} \left( e^{Z^T, k\tau^{1/8}} - 1 \right)^+ \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi_{\tau, k}(u) \overline{C}_{k, \tau}(u) \, du, \]

‘conclude’ using (1) and a control of the tails

\[ \left| \int_{|u| > 1/\sqrt{\varepsilon}} \Phi_{\tau, k}(u) \overline{C}_{\varepsilon, k}(u) \, du \right| = \mathcal{O}(e^{-\gamma/\varepsilon}). \]
Getting some intuition: Gärtner-Ellis theorem

Rescaled cumulant generating function \( (h \) continuous with \( \lim_{\tau \downarrow 0} h(\tau) = 0) \):

\[
\Lambda_\tau(u, h) := h(\tau) \log \mathbb{E} \left( S_\tau^{u/h(\tau)} \right) = h(\tau) \log \mathbb{E} \left( e^{(u/h(\tau))X_\tau} \right), \quad u \in D_{\tau,h} \subset \mathbb{R}.
\]

**Theorem (Gärtner-Ellis)**

If \( \Lambda(u) := \lim_{\tau \downarrow 0} \Lambda_\tau(u, h) \) exists in \( \mathbb{R} \) for \( u \in D_0 \), and \( \Lambda \) strictly convex and differentiable on \( D_0^o \), with \( \lim_{u \in \partial D_0} \Lambda'(u) = +\infty \), then \( (X_\tau)_{\tau > 0} \) satisfies a large deviations principle (LDP) (with speed \( h(\tau) \)) as \( \tau \downarrow 0 \):

\[
P(X_\tau \in A) \sim \exp \left( -\frac{1}{h(\tau)} \inf \{ \Lambda^*(x) : x \in A \} \right), \quad A \subset \mathbb{R}.
\]

**Lemma:** For diffusions (Black-Scholes, stochastic volatility,...), exp-Lévy, \( h(\tau) \equiv \tau \).
Getting some intuition: Gärtner-Ellis theorem

Rescaled cumulant generating function ($h$ continuous with $\lim_{\tau \downarrow 0} h(\tau) = 0$):

$$\Lambda_\tau(u, h) := h(\tau) \log \mathbb{E} \left( S_\tau^{u/h(\tau)} \right) = h(\tau) \log \mathbb{E} \left( e^{(u/h(\tau))X_\tau} \right), \quad u \in D_\tau, h \subset \mathbb{R}.$$  

**Theorem (Gärtner-Ellis)**

If $\Lambda(u) := \lim_{\tau \downarrow 0} \Lambda_\tau(u, h)$ exists in $\mathbb{R}$ for $u \in D_0$, and $\Lambda$ strictly convex and differentiable on $D_0^\circ$, with $\lim_{u \in \partial D_0} \Lambda'(u) = +\infty$, then $(X_\tau)_{\tau > 0}$ satisfies a large deviations principle (LDP) (with speed $h(\tau)$) as $\tau \downarrow 0$:

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**Lemma**: For diffusions (Black-Scholes, stochastic volatility,...), exp-Lévy, $h(\tau) \equiv \tau$.

**Forward problem**: $S$: Heston model, and $\Lambda^{(t)}_\tau$ the rescaled forward cgf, then

**Lemma**: If $h(\tau) \equiv \sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda^{(t)}_\tau(u, h(\tau)) = 0$, for $u \in (\underline{u}, \bar{u})$ and $\infty$ otherwise;

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Asymptotics of forward implied volatility