Local Correlation with Local Vol and Stochastic Vol: Towards Correlation dynamics?

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10th January 2014
Outline

1. Local Correlation: where are we?
2. PnL equation
3. Observe correlation
   - Evidence of Correlation Skew
4. Model correlation?
   - Introduce Decorrelation
   - New Methods in Finance
   - Local Formulae: Derivate Market Information
5. Why basket local correlation?
6. Calibration results: Local Volatility
7. Extension to Stochastic Volatility
   - Need to introduce specific parametrization
   - Decorrelation with Multi-Underlying Stochastic Volatility
   - Usual values of correl between vols
8. Focus on correlation products
9. Main conclusions
10. References
Recent (or less recent) developments in local correlation

- Avellaneda: local formula + approximation
- Reghai: based on fixed point algorithm, but slow convergence (cf. based on implied vols)
- Langnau: pathwise equality of covariance to calibrate local correl, too many constraints? (cf. sufficient but not necessary condition)
- Sbai-Jourdain: top-down approach (insert index in stock diffusion) instead of usual bottom up, but issues since introduces historical parameter $\beta$
- Piterbarg: markovian projection, calibration based on approximations (not specific to correlation)
- Guyon-Henry-Labordere: "Particle Methods" (not specific to correlation)

Our approach = similar to Particle Methods, but method slightly differs for specific points.
"Overomega" Definition

Banks usually short correlation (cf. sell basket calls/puts, sell WO Calls,...) => need to overprice Correlation.
Constraint : needs to remain PSD!
Solution : use the convexity for space of correlation matrix (standard, also used in shrinkage methods)

We introduce "Overomega" (not a standard notation) :

\[ \rho_{i,j}^{Pricing} = (1 - \omega) \rho_{i,j}^{Histo} + \omega \]

Generally \( \omega \approx 15\% \).
Used to give conservative prices.

Remember : not always true (sell spread options,...)!
Conservative pricing : Need to choose adapted target matrix (cf. crossed gamma sign), with \( PricingMatrix = (1 - \omega)InitMatrix + \omega TargetMatrix \)

But Issues when Crossed gammas change sign locally (\( \Rightarrow \) uncertain correlation pricing)
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Structured Equity Research (HSBC)
Why does correlation matters: PnL Equation

Consider a product with value $P$ that we buy. Pricing equation

$$ r_t P = \frac{\partial P}{\partial t} + \sum_i \frac{\partial P}{\partial x_i} r_t x_i + \sum_{i,j} \frac{1}{2} \frac{\partial^2 P}{\partial x_i \partial x_j} \rho_{i,j} \sigma_i \sigma_j x_i x_j $$

PnL equation (integrated = "tracking error"):

$$ \Delta P - r P \Delta t - \sum_i \frac{\partial P}{\partial x_i} (\Delta S_i - r S_i \Delta t) = \frac{\partial P}{\partial t} \Delta t + \sum_i \frac{\partial P}{\partial x_i} \Delta S_i + \sum_{i,j} \frac{1}{2} \frac{\partial^2 P}{\partial x_i \partial x_j} \Delta S_i \Delta S_j - r P \Delta t - \sum_i \frac{\partial P}{\partial x_i} (\Delta S_i - r S_i \Delta t) $$

$$ = \frac{1}{2} \sum_i \frac{\partial^2 P}{\partial x_i^2} (\Delta S_i^2 - \sigma_i^2 (S_i)^2 \Delta t) + \sum_{i,j} \frac{\partial^2 P}{\partial x_i \partial x_j} (\Delta S_i \Delta S_j - \rho_{i,j} \sigma_i \sigma_j S_i S_j \Delta t) $$

Analysis:

- Link between Cegas (Correlation Greeks) and Crossed Gammas.
- Short Crossed Gamma and correlated movement, loses money
- Need to use a model with a theta coherent with these crossed gammas
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What is correlation?

Correlation not a "clean" quantity, more adequate quantity = covariance.

Correlation = for given vol and given covariance, way to introduce link between brownians (generally, "Gaussian copula")

Example of issue: correlation can be more than 1 due to time zones (Bergomi) or other (model) reasons. No way (that I know of) to deal with this issue in Monte-Carlo. (and seems to present numerical issues in PDE and Fourier)

Natural question: what is Implied Correlation?

**Implied Vol**

Rebonato: "wrong number to put in the wrong formula to get the right price"

**Implied Correlation**

"wrong number to put in the wrong pricer given a wrong volatility model to get the right price"
Observe correlation

Implied Correlation Data

Example: ICJ/JCJ/KCJ rotating indexes.

Currently: JCJ (Jan. 2014) or KCJ Index (Jan. 2015). Different issues

- Reference Vol Model = Black-Scholes
- Based on approximate formula (most likely path)
- Implied Volatility = for stocks, ATM Spot Implied Vol and not ATMF implied Vols
- Based on only 50 underlyings of SP500 (liquidity issues)
Interpolated 1Y Implied Correlation from ICJ/JCJ/KCJ (since 2007). Evolution.

Figure: Evidence of Correlation Skew
Evidence of correlation Skew based on Historical Data

Interpolated 1Y Implied Correlation from ICJ/JCJ/KCJ (since 2007)

Figure: Evidence of Correlation Skew
Evidence based on Implied Data(1)

![Graph showing Index versus Basket Smile: 1Y smile 06/03/2013 (SMI)](image)

**Figure:** Basket Smile versus Index Smile: SMI case

Consequence: market expects more correlation on the downside, and less on the upside.
Evidence based on Implied Data(2)

**Figure:** Index Implied Correlation

\[ \Rightarrow \text{Correlation increases when basket decreases.} \]

Note: Here, Overomega skew and not Correlation Skew (ratio \(1 - \bar{\rho}\) between both)
Rationale behind correlation skew

Correlation Skew = market evidence.
Main reasons:

- Law of demand and supply: more buyers on the downside (protection)
- Systemic risk: big downward moves, risk linked to economy, all stocks impacted

Upside: generally decreases, but (sometimes) systemic "rescue". When good news concerning the economy (rates decrease, central bank actions,...), all stocks impacted (and correlation increases).
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The purpose of modelling correlation

Different situations:
Our focus: liquid basket options
However, no real hedge strategy since basket composition changes:

⇒ essentially Macro Management Tool.

Steps:
1. Decorrelate initial correlation matrix
2. Write Local Formula linking two different models
3. Use fixed-point algorithm (or particle method) to calibrate
Decorrelation Step

Ideas:
- Decorrelate initial correlation matrix
- Use parametric local overomega to recorrelate the matrix

Decorrelation:

$$\rho_{i,j}^D = (1 - \omega_1)\rho_{i,j}^H + \omega_1 \text{ with } \omega_1 < 0$$

$$\iff \rho_{i,j}^H = (1 - \omega_0)\rho_{i,j}^D + \omega_0 \text{ with } \omega_0 = \frac{\omega_1}{\omega_1 - 1}$$

In practice, maximize $|\omega_1|$ so that matrix remains PSD and with positive correlation.
Foreword

Standard Models are simpler to handle with local vol, local correl adjustments:

- Fixed Point algorithm (Reghai) +
- Local Formulae (Dupire) +
- Numerical Evaluation of conditional expectations (not specifically linked to finance) =
- Local fixed-point methods (particular case for explicit schemes with one iteration = Particle method)

Fixed Point problem: contracting function(?) on a space of stochastic processes. Existence still needs to be solved theoretically.
Remember: how to prove Dupire’s formula?

Idea: Derive Market Information/Observables

\[
dS_t = (r_t S_t - Q_t - q_t S_t) dt + \sigma(t, S_t) S_t dW_t
\]

so that (undiscounted calls) \(dC\)

\[
d\mathbb{E}^Q(S_t - K)^+ = \frac{\partial C}{\partial t} dt
\]

\[
= \mathbb{E}^Q d(S_t - K)^+
\]

\[
= \mathbb{E}^Q (dS_t 1_{S_t > K} + \frac{1}{2} d < S > t \delta_{S_t=K})
\]

\[
= \mathbb{E}^Q ((r_t - q_t)S_t 1_{S_t > K} - Q_t 1_{S_t > K} + \frac{1}{2} \sigma(t, K)^2 K^2 \delta_{S_t=K}) dt
\]

But: \(\frac{\partial C}{\partial K}\)

\[
= -\mathbb{E}^Q (1_{S_t > K})
\]

Or: \(C - K \frac{\partial C}{\partial K}\)

\[
= \mathbb{E}^Q (S_t 1_{S_t > K})
\]

And: \(\frac{\partial^2 C}{\partial K^2}\)

\[
= \mathbb{E}^Q (\delta_{S_t=K})
\]

So that: \(\frac{\partial C}{\partial t}\)

\[
= (r_t - q_t)(C - K \frac{\partial C}{\partial K}) + Q_t \frac{\partial C}{\partial K} + \frac{1}{2} \sigma(t, K)^2 K^2 \frac{\partial^2 C}{\partial K^2}
\]

And finally: \(\sigma(t, K)^2\)

\[
= \frac{\frac{\partial C}{\partial t} - (r_t - q_t)(C - K \frac{\partial C}{\partial K}) - Q_t \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}
\]
Our framework = Reghai’s

Local Correlation introduced through the use of an overomega approach.

What is Overomega ? \( \rho_{i,j}^{Pricing} = (1 - \omega)\rho_{i,j}^{Historical} + \omega \)

First Model = Simple Local Vol Model with continuous dividends (mix of prop and cash dividends).

\[
dS_t^i = (r_t S_t^i - Q_t^i - q_t^i S_t^i)dt + \sigma(t, S_t^i) S_t^i \sqrt{1 - \omega(t, I_t^S)} dW_t^i + \sqrt{\omega(t, I_t^S)} dW_t^\perp
\]

with \( Q_t^i \) and \( q_t^i \) deterministic and:

\[
I_t^S = \sum_{i=1}^{N} w_i S_t^i
\]

\[
< dW_t^i, dW_t^j > = \rho_0^{ij} dt
\]

\[
< dW_t^i, dW_t^\perp > = 0 \forall i
\]
Local Correlation formula (general case)

Second Model = simple local vol model written on the index (continuous divs):

\[ dl_t = (r_t l_t - Q_t - q_t l_t) dt + l_t \sigma(t, l_t) dW_t \]

with \( Q_t \) and \( q_t \) deterministic.

Same Basket Call prices in both models (Specific set of \( w_i \)):

\[ C(K, t) = \mathbb{E}^Q(\exp(-\int_0^t r_s ds)(l_t - K)^+) \]

\[ = \mathbb{E}^Q(\exp(-\int_0^t r_s ds)(l_t^S - K)^+) \forall t, K \]

but:

\[ \frac{\partial C}{\partial t} dt = \mathbb{E}^Q(\exp(-\int_0^t r_s ds)((dl_t^S - r_t(l_t^S - K) dt)1_{l_t^S > K} + \frac{1}{2} d < l_t^S > t \delta_{l_t^S = K})) \text{ in basket model} \]

\[ \frac{\partial C}{\partial t} dt = \mathbb{E}^Q(\exp(-\int_0^t r_s ds)((dl_t - r_t(l_t - K) dt)1_{l_t > K} + \frac{1}{2} d < l_t > t \delta_{l_t = K})) \text{ in index model} \]
Local Correlation formula (2)

\[
\mathbb{E}^{Q_t}(\sum_i w_i(Q^i_t + q^i_t S^i_t) + r_t K dt) 1_{I^S_t > K} + \frac{1}{2} d < I^S_t > t \delta_{I^S_t = K}) = \mathbb{E}^{Q_t}(\sum_i w_i q^i_t S^i_t) 1_{I^S_t > K} + \frac{1}{2} d < I^S_t > t \delta_{I^S_t = K})
\]

but:

\[
\mathbb{E}^{Q_t}(d < I^S_t > t \delta_{I^S_t = K}) = \mathbb{E}^{Q_t}(d < I^S_t > t | I^S_t = K) \frac{\partial^2 C}{\partial K^2}
\]

and also:

\[
\mathbb{E}^{Q_t}(I^S_t > K) = \mathbb{E}^{Q_t}(I^S_t > K) = \frac{1}{B(0, t)} (C - K \frac{\partial C}{\partial K})
\]

\[
\mathbb{E}^{Q_t}(1_{I^S_t > K}) = \mathbb{E}^{Q_t}(1_{I^S_t > K}) = \frac{1}{B(0, t)} (- \frac{\partial C}{\partial K})
\]

Condition on the forward \((K = 0)\):

\[
Q_t = \sum_i w_i Q^i_t
\]
\[
q_t = \frac{\mathbb{E}^{Q_t}(\sum_i w_i q^i_t S^i_t)}{\mathbb{E}^{Q_t}(I_t)}
\]
Local Correlation formula (3)

\[
\omega(t, K) = \left( K^2 \sigma(t, K)^2 + \frac{2}{\partial K} \frac{\partial C}{\partial K} \left( \frac{\mathbb{E}^Q((q_t l_t - r_t K)1_{t > K})}{\mathbb{E}^Q(1_{t > K})} - \frac{\mathbb{E}^Q(\sum_i w_i q^i S_t - r_t K)1_{t > K}}{\mathbb{E}^Q(1_{t > K})} \right) \right) \frac{\mathbb{E}^Q(\sum_{i,j} w_i w_j S_{t}^i S_{t}^j \sigma(t, S_t^i) \sigma(t, S_t^j)(1 - \rho_{i,j})|I_t^S = K)}{\mathbb{E}^Q(\sum_{i,j} w_i w_j S_{t}^i S_{t}^j \sigma(t, S_t^i) \sigma(t, S_t^j) \rho_{i,j} |I_t^S = K)}
\]
Local Correlation formula: simplest formula

Particular cases: no dividends, deterministic interest rates

\[
\omega(t, K) = \frac{K^2 \sigma(t, K)^2 - \mathbb{E}^Q \left( \sum_{i,j} w_i w_j S_t^i S_t^j \sigma(t, S_t^i) \sigma(t, S_t^j) \rho_{i,j}^0 | I_t^S = K \right)}{\mathbb{E}^Q \left( \sum_{i,j} w_i w_j S_t^i S_t^j \sigma(t, S_t^i) \sigma(t, S_t^j) (1 - \rho_{i,j}^0) | I_t^S = K \right)}
\]

Dupire/Avellaneda/Piterbarg/Guyon-PHL formula.

Case where constant vol and null correlation:

\[
\omega = \frac{\sigma_i^2 - \frac{1}{N} \sigma_S^2}{\sigma_S^2 \left(1 - \frac{1}{N}\right)}
\]

Well known formula: see Bossu for example.

Idea = Depends on covariance: \(\frac{\text{Implied} - \text{Minimum}}{\text{Maximum} - \text{Minimum}}\)
Local Correlation formula: focus on dividends

\[
\frac{\mathbb{E}^{Q_t}((q_t l_t - r_t K)1_{l_t > K})}{\mathbb{E}^{Q_t}(1_{l_t > K})} - \frac{\mathbb{E}^{Q_t}((\sum_i w_i q_i S^i_t - r_t K)1_{l^S_t > K})}{\mathbb{E}^{Q_t}(1_{l^S_t > K})}
\]

Stochastic rate term + Dividend term.

Deterministic interest rates: first term vanishes since \( r_t \) in factor and \( \mathbb{E}^{Q_t}(1_{l_t > K}) = \mathbb{E}^{Q_t}(1_{l^S_t > K}) = \frac{1}{\mathbb{B}(0, t)} ( - \frac{\partial C}{\partial K} ) \)

Residual term linked to dividends: cf. no arbitrage condition in case of discrete dividends:

\[
\mathbb{E}^{Q_t}((l_t - K)^+) - \mathbb{E}^{Q_t}((l_{t-} - K)^+) \simeq \mathbb{E}^{Q_t}((l_{t-})1_{l_t > K}) \\
\mathbb{E}^{Q_t}((l^S_t - K)^+) - \mathbb{E}^{Q_t}((l^S_{t-} - K)^+) \simeq \mathbb{E}^{Q_t}((l^S_{t-})1_{l^S_t > K})
\]

but:

\[
l_t - l_{t-} = -(Q_t + q_t l_{t-}) \\
l^S_t - l^S_{t-} = \sum_i -w_i(Q^i_t + q_t S^i_{t-})
\]

leads to (first order in dividend level):

\[
\mathbb{E}^{Q_t}(q_t l_t 1_{l_t > K}) = \mathbb{E}^{Q_t}(\sum_i w_i q^i_t S^i_t 1_{l^S_t > K})
\]
Local Correlation formula: focus on dividends (2)

If discrete dividends: impossible to achieve for each $K$ if $q_t$ constant (except in particular cases: null volatility or $q_t = q^i_t \forall i$)

$\implies$ two models are generally inconsistent.

$\implies$ Need to use continuous div model

One more derivation in $K +$ same density $(\frac{\partial^2 C}{\partial K^2})$ give:

$$
\mathbb{E}^Q(t)(q_t | I_t = K) = \mathbb{E}^Q(t)\left(\sum_i w_i q_t^i S_t^i | I_t^S = K\right)
$$

cf. Markovian projection: sufficient but not necessary condition

Other possible conditions:

$$
\begin{aligned}
q_t &= \frac{\mathbb{E}^Q(\sum_i w_i q_t^i S_t^i)}{\mathbb{E}^Q(t)} \\
\omega(t, K) &= \left(\kappa^2 \sigma(t, K)^2 - \frac{2(C - K \frac{\partial C}{\partial K})}{\frac{\partial^2 C}{\partial K^2}} \left(q_t - \mathbb{E}^Q(\sum_i w_i q_t^i S_t^i | I_t^S > K)\right)\right) - \mathbb{E}^Q(\sum_{i,j} w_i w_j S_t^i S_t^j \sigma(t, S_t^i) \sigma(t, S_t^j) \rho_{i,j}^0 | I_t^S = K)
\end{aligned}
$$

Comments:

- $\omega$ helps recover from the generated error
- in practice, prop divs smile correction can be neglected
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Why basket local correlation?

Other possible local correlations!

Reghai: Consider WO, BO, Rainbow local correlation to handle chewing-gum effect

Example: Worst Of Local Correlation (for WO products)
Two models: Worst Of Model and standard model with WO local correlation

WO Model:
\[
\frac{dWO_t}{WO_t} = (r_t - q_t)dt + \sigma(t, WO_t)dW_t
\]

Standard Model:
\[
\frac{dS_t^i}{S_t^i} = (r_t - q_t^i)dt + \sigma^i(t, S_t^i)(\sqrt{1 - \omega(t, \tilde{WO}_t)}dW_t^i + \sqrt{\omega(t, \tilde{WO}_t)}dW_t^⊥)
\]

with: \( \tilde{WO}_t = \min_i \left( \frac{S_t^i}{S_{t_0}^i} \right) \)

and: \( <dW_t^i, dW_t^j> = \rho_{i,j}^0(t)dt \)
Why basket local correlation?

Worst Of Correlation (2)

Derive WO Calls in both models (calculation a little heavy):

WO model: \[
\frac{\partial C_{WO}}{\partial t} = -r_tC_{WO} + (r_t - q_t)(C_{WO} - K \frac{\partial C_{WO}}{\partial K}) + \frac{1}{2} K^2 \sigma^2(t, K) \frac{\partial^2 C_{WO}}{\partial K^2}
\]

WO local correl model: \[
\frac{\partial C_{WO}}{\partial t} = -r_tC_{WO} + (r_t - q_t(K))(C_{WO} - K \frac{\partial C_{WO}}{\partial K}) + \frac{1}{2} K^2 \mathbb{E}^q(\sigma_{WO}(t, K)^2 | \text{WO} = K) \frac{\partial^2 C_{WO}}{\partial K^2} - \frac{1}{2} \sum_{i>j} \mathbb{E}^q(d < S_t^i, S_t^j > + d < S_t^i, S_t^j > - 2d < S_t^i, S_t^j >) \delta_{S_t^i = S_t^j, 1, \text{WO}_t > K, \text{WO}_t = S_t^i, 1} - \frac{1}{2} \sum_{i>j} \mathbb{E}^q(2(1 - \rho_{i,j}^0(t))\sigma_i(t, K)\sigma_j(t, K) \delta_{S_t^i = S_t^j, 1, \text{WO}_t > K, \text{WO}_t = S_t^i, 1} - \frac{1}{2} \sum_{i>j} \mathbb{E}^q(2\sigma^2_i(t, K) - \sigma^2_j(t, K)) \delta_{S_t^i = S_t^j, 1, \text{WO}_t > K, \text{WO}_t = S_t^i, 1}) \]

with: \[q_t^{WO}(K) = \frac{\mathbb{E}^q(q_t^{WO} \text{WO} > K)}{\mathbb{E}^q(\text{WO} > K)}
\]

Condition on WO Forward: \[q_t = q_t^{WO}(0)
\]

If \[q_t^{WO}(K) = q_t \] (not true in general, else add corrective term to overomega like in basket formula) and \[\rho_{i,j}(t, K) = \rho_{i,j}^0(t) + \omega(t, K)(1 - \rho_{i,j}^0(t)), \] one more \(K\) derivation gives:

\[\omega(t, K) = \frac{\frac{\partial}{\partial K} \left( K^2 \frac{\partial^2 C_{WO}}{\partial K^2} \right) \left( \mathbb{E}^q(\sigma_{WO}(t, K)^2 | \text{WO} = K) - \sigma^2(t, K) \right)}{K^2 \sum_{i>j} \mathbb{E}^q(2(1 - \rho_{i,j}^0(t))\sigma_i(t, K)\sigma_j(t, K) \delta_{S_t^i = S_t^j, 1, \text{WO}_t > K, \text{WO}_t = S_t^i, 1})} - \frac{1}{2} \sum_{i>j} \mathbb{E}^q(2\sigma^2_i(t, K) - \sigma^2_j(t, K)) \delta_{S_t^i = S_t^j, 1, \text{WO}_t > K, \text{WO}_t = S_t^i, 1}) \]

\[\text{HSBC} \times \]

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Why basket local correlation?

**Worst Of Correlation (3)**

Two important quantities:

- **Switching Local Time**: \( \delta_{t=\bar{t}} \)
- **Local Dispersion**:
  \[
  \frac{d < S_t^1, S_t^2 > + d < S_t^1, S_t^2 > - 2d < S_t^1, S_t^2 >}{dt} = \frac{d < S_t^1 - S_t^2, S_t^1 - S_t^2 >}{dt} = (\sigma^i S^i)^2 + (\sigma^j S^j)^2 - 2\rho_{i,j} \sigma^i \sigma^j S^i S^j
  \]

Local Dispersion = short correl, long volatility, positive quantity

cf. \(-2\rho_{i,j} \sigma^i \sigma^j S^i S^j + (\sigma^i S^i)^2 + (\sigma^j S^j)^2 = (\sigma^i S^i - \sigma^j S^j)^2 + 2(1 - \rho_{i,j})\sigma^i \sigma^j S^i S^j\)

Note: local dispersion in Spread Option Local equation:

\[
\frac{\partial C^{Spread}}{\partial t} = -r_t C^{Spread} + (r_t - q_t^{Spread})(C^{Spread} - K \frac{\partial C^{Spread}}{\partial K})
+ \frac{1}{2} \mathbb{E}^Q \left( \frac{d < S_t^1, S_t^2 > + d < S_t^2, S_t^2 > - 2d < S_t^1, S_t^2 >}{dt} \right) \delta_{Spread=K}
\]

Remember also Margrabe formula: \( \sigma = \sqrt{(\sigma^i)^2 + (\sigma^j)^2 - 2\rho_{i,j} \sigma^i \sigma^j} \)

\(\implies\) WO Call short disp product, spread option long disp product.
Why basket local correlation?

Local Correlation Models Limits

WO local Correlation:
- no real observable smile for WO vanillas
- Dynamic issue: only valid at inception (local vol -and forward- of WO model should change dynamically but how?)
- more complex and less stable numerically
- not much financial sense: how to infer a historical WO local correlation skew?
- but "chewing gum" effect

Basket local Correlation:
- not many observables but more precise idea of hypothetic smile
- Dynamic issue: only valid at inception (local vol of basket with changed weights should change dynamically but how?)
- simple and stable numerically
- financial sense (cf. historical observations)

=> we will study Basket Local Correlation.
Discussion : $\omega \in [0; 1]$ ?


- Not theoretically (no static arbitrage)
- True in practice if $\rho_{i,j}^0$ enough low (Ex: $\rho_{i,j}^0 = 0 \forall i,j$)
- Explanation ?
  - Possible to infer a positive implied correlation $\omega^l(K, T)$ for a standard model if $\rho_{i,j}^0 = 0$ for ex.
  - Gatheral-like formula :
    $$\rho_{i,j}^l(T, m)\sigma_{i}^l(T, m)\sigma_{j}^l(T, m) \simeq \frac{1}{T} \int_{0}^{T} \rho_{i,j}^L(t, \frac{mt}{T})\sigma_{i}^L(t, \frac{mt}{T})\sigma_{j}^L(t, \frac{mt}{T})dt$$
- Introduction of drift (continuous dividends) still OK.
Why basket local correlation?

Parametric Regression

Need to estimate:

\[ E^Q \left( \sum_{i,j} w_i w_j S_t^i S_t^j \sigma(t, S_t^i) \sigma(t, S_t^j) \rho_{i,j}^0 \mid I_t^S = K \right) \]

\[ E^Q \left( \sum_{i,j} w_i w_j S_t^i S_t^j \sigma(t, S_t^i) \sigma(t, S_t^j) \mid I_t^S = K \right) \]

What do they look like?

Figure: Variable To Explain versus Basket

Interest: instead of non parametric regression, natural regression on 
\((1, B, B^2, \ldots, B^p)\) can also be used. Proves to be stable and complexity in \(O(Np)\)
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Results

Application to Eurostoxx smile.
Only two iterations that need 2000 simulations each: quick calibration. Here, $\rho_{i,j}^0 = 0$.

Figure: Fitting the Index Smile using Correlation Skew
Local Correlation Shape

**Figure:** Local Correlation Smile

**Figure:** ATM Local Correlation Skew Term Structure Eurostoxx (05/04/2013)
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Extension to Stochastic Volatility framework

Chosen volatility model = (continuous) Bergomi model:

\[
\frac{dS_t^i}{S_t^i} = \sigma(t, S_t^i)\sqrt{\xi_{t^i,t}^i}\left(\sqrt{1 - \omega(t, I_t^S)}dW_t^i + \sqrt{\omega(t, I_t^S)}dW_t^\perp\right)
\]

or

\[
\frac{dS_t^i}{S_t^i} = \sqrt{\xi_{t^i,t}^i}d\tilde{W}_t^i
\]

with:

\[
d\tilde{W}_t^i = \sqrt{1 - \omega(t, I_t^S)}dW_t^i + \sqrt{\omega(t, I_t^S)}dW_t^\perp
\]

\[
\frac{d\xi_{t^i,T}^i}{\xi_{t^i,T}^i} = \sigma_S^i \exp(-\kappa_S^i(T - t))dW_t^{i,S} + \sigma_L^i \exp(-\kappa_L^i(T - t))dW_t^{i,L}
\]

with:

\[
\xi_{t^i,T}^i = \mathbb{E}^Q(V_t^i | F_t)
\]

\[
<dW_t^i, dW_t^j> = \rho_{i,j}^0 dt
\]

\[
dW_t^{i,S} = \rho_S^i d\tilde{W}_t^i + \sqrt{1 - (\rho_S^i)^2(\alpha_t dW_t^i + \sqrt{1 - (\alpha)^2}dW_t^{i,S})}
\]

\[
dW_t^{i,L} = \rho_L^i d\tilde{W}_t^i + \sqrt{1 - (\rho_L^i)^2(\beta_t dW_t^i + \sqrt{1 - (\beta)^2}dW_t^{i,L})}
\]

\[
<dW_t^{i,S}, dW_t^{i,L}> = \rho_{SL}^i dt
\]

Comments:

- 3N + 2 brownians required \((NW, NW^L, NW^S, Z, W^\perp)\)
- Parametrization maintains mono underlying volatility skew due to stochastic vol.
Correlation Structure

Complicated PSD conditions? No, because direct use of brownians
\[ \implies \text{no cholesky} \implies \text{computation time gain} \]
But: \( \alpha \) and \( \beta_i \in [-1; 1] \)
Notice that:
\[ \beta_i = \frac{1}{\alpha} \frac{\rho_{SL}^i - \rho_S^i \rho_L^i}{\sqrt{1 - (\rho_S^i)^2} \sqrt{1 - (\rho_L^i)^2}} \]
but:
\[ 1 + 2 \rho_S^i \rho_L^i \rho_{SL}^i - (\rho_S^i)^2 - (\rho_L^i)^2 - (\rho_{SL}^i)^2 \geq 0 \text{ (cf. PSD for each underlying)} \]
\[ \iff \left( \frac{\rho_{SL}^i - \rho_S^i \rho_L^i}{\sqrt{1 - (\rho_S^i)^2} \sqrt{1 - (\rho_L^i)^2}} \right)^2 \leq 1 \]
\[ \iff \alpha \beta_i \in [-1; 1] \]

Comments:
- \( \alpha = 1 \): OK
- \( \rho_{SL}^i = \rho_S^i \rho_L^i \): OK for any \( \alpha \)
- Remember \( \alpha = 1 \): basket prices closest with LV and SV
Decorrelation Effect

Observation = "Decorrelation Effect with Stochastic Volatility"

If $\forall i, V_{i}^{Sto}(K) = V_{i}^{Loc}(K)$ then $V_{bskt}^{Sto}(K) \leq V_{bskt}^{Loc}(K)$ where $V=$ Call or Put

Why? Heuristic:

\[
\mathbb{E}^{Q}\left(\sum_{i,j} w_{i}w_{j}\sqrt{V_{i}}\sqrt{V_{j}}s_{i}^{t}s_{j}^{t}\rho_{i,j}\mid \sum_{i} w_{i}s_{i}^{t} = \sum_{i} w_{i}F_{i}^{t}\right) \approx \mathbb{E}^{Q}\left(\sum_{i,j} \frac{1}{N^{2}} \sqrt{V_{i}}\sqrt{V_{j}}\rho_{i,j}\mid \forall k, s_{k}^{t} = F_{k}^{t}\right)
\]

and $\mathbb{E}^{Q}\left(\sum_{i,j} w_{i}w_{j}(t, s_{i}^{t})\sigma_{i}(t, s_{i}^{t})(t, s_{i}^{t})s_{i}^{t}s_{j}^{t}\rho_{i,j}\mid \sum_{i} w_{i}s_{i}^{t} = \sum_{i} w_{i}F_{i}^{t}\right) \approx \sum_{i,j} \frac{1}{N^{2}} \sigma_{i}(t, F_{i}^{t})\sigma_{j}(t, F_{j}^{t})\rho_{i,j}$

Cauchy-Schwarz: $\mathbb{E}^{Q}\left(\sqrt{V_{i}}\sqrt{V_{j}}|s_{i}^{t} = F_{i}^{t}, s_{j}^{t} = F_{j}^{t}\right) \lesssim \sqrt{\mathbb{E}^{Q}(V_{i}|s_{i}^{t} = F_{i}^{t})\mathbb{E}^{Q}(V_{j}|s_{j}^{t} = F_{j}^{t})}$

or $\mathbb{E}^{Q}\left(\sqrt{V_{i}}\sqrt{V_{j}}|s_{i}^{t} = F_{i}^{t}, s_{j}^{t} = F_{j}^{t}\right) \lesssim \sigma_{i}(t, F_{i}^{t})\sigma_{j}(t, F_{j}^{t})$

so that $\mathbb{E}^{Q}\left(\sum_{i,j} w_{i}w_{j}\sqrt{V_{i}}\sqrt{V_{j}}s_{i}^{t}s_{j}^{t}\rho_{i,j}\mid \sum_{i} w_{i}s_{i}^{t} = \sum_{i} w_{i}F_{i}^{t}\right) \lesssim \mathbb{E}^{Q}\left(\sum_{i,j} w_{i}w_{j}(t, s_{i}^{t})\sigma_{i}(t, s_{i}^{t})(t, s_{i}^{t})s_{i}^{t}s_{j}^{t}\rho_{i,j}\mid \sum_{i} w_{i}s_{i}^{t} = \sum_{i} w_{i}F_{i}^{t}\right)$
Decorrelation Effect

The decorrelation effect depends a great deal on the value of \( \alpha \) (correlation between vols) and a little on the size of the basket.

**Figure:** Decorrelation Effect depending on basket size
Historical value of implied volatilities

Figure: Historical correlation between vols for main indices

⇒ High level of correlation between vols.

Link between Correl and $\alpha$? One factor case:

$$\rho_{V_i, V_j} = \rho_{S_i, V_i} \rho_{S_j, V_j} + \alpha^2 \sqrt{1 - \rho_{S_i, V_i}^2} \sqrt{1 - \rho_{S_j, V_j}^2}$$

Standard values: $\alpha \approx 1$ or $\alpha > 1 \implies$ need for level correction.
Calibration formula

Calibration (no div case) using fixed point algorithm with parametric (polynomial in basket moneyness) or non-parametric regression.

Formula:

\[ \omega^{(n+1)}(t, K) = \frac{K^2 \sigma(t, K)^2 - \mathbb{E}^Q(\sum_{i,j} w_i w_j S_{t_i}^{i,(n)} S_{t_j}^{j,(n)} \sigma(t, S_{t_i}^{i,(n)}) \sigma(t, S_{t_j}^{j,(n)}) \sqrt{\xi_t^{i,t}} \rho_{i,j}^{0} \mid I_{t}^{S,(n)} = K)}{\mathbb{E}^Q(\sum_{i,j} w_i w_j S_{t_i}^{i,(n)} S_{t_j}^{j,(n)} \sigma(t, S_{t_i}^{i,(n)}) \sigma(t, S_{t_j}^{j,(n)}) (1 - \rho_{i,j}^{0}) \mid I_{t}^{S,(n)} = K)} = \frac{K^2 \sigma(t, K)^2 - f^{(n)}(t, K)}{g^{(n)}(t, K) - f^{(n)}(t, K)} \]
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## Analysis of correlation product prices in different models

World Basket as of 05/04/2013. Maturity = 1Y. Strikes in Forward Basket Moneyness.

<table>
<thead>
<tr>
<th>Product/Model</th>
<th>Without CS</th>
<th>With CS</th>
<th>With CS and SV / $\alpha = 1$</th>
<th>With CS and SV / $\alpha = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward WO</td>
<td>90.15%</td>
<td>90.25%</td>
<td>90.48%</td>
<td>90.52%</td>
</tr>
<tr>
<td>WO Call 90</td>
<td>8.73%</td>
<td>8.45%</td>
<td>8.53%</td>
<td>8.54%</td>
</tr>
<tr>
<td>WO Call 95</td>
<td>6.27%</td>
<td>5.97%</td>
<td>6.00%</td>
<td>6.01%</td>
</tr>
<tr>
<td>WO Call 100</td>
<td>4.28%</td>
<td>3.98%</td>
<td>3.98%</td>
<td>3.98%</td>
</tr>
<tr>
<td>WO Call 105</td>
<td>2.76%</td>
<td>2.49%</td>
<td>2.47%</td>
<td>2.47%</td>
</tr>
<tr>
<td>WO Call 110</td>
<td>1.66%</td>
<td>1.14%</td>
<td>1.42%</td>
<td>1.42%</td>
</tr>
<tr>
<td>Forward BO</td>
<td>109.49%</td>
<td>109.27%</td>
<td>109.34%</td>
<td>109.32%</td>
</tr>
<tr>
<td>BO Put 90</td>
<td>1.88%</td>
<td>2.34%</td>
<td>2.28%</td>
<td>2.32%</td>
</tr>
<tr>
<td>BO Put 95</td>
<td>2.81%</td>
<td>3.32%</td>
<td>3.24%</td>
<td>3.28%</td>
</tr>
<tr>
<td>BO Put 100</td>
<td>4.11%</td>
<td>4.64%</td>
<td>4.55%</td>
<td>4.58%</td>
</tr>
<tr>
<td>BO Put 105</td>
<td>5.84%</td>
<td>6.38%</td>
<td>6.27%</td>
<td>6.30%</td>
</tr>
<tr>
<td>BO Put 110</td>
<td>8.09%</td>
<td>8.62%</td>
<td>8.48%</td>
<td>8.51%</td>
</tr>
</tbody>
</table>

Spread Option Eurostoxx versus SP500. Strikes in Forward Spread Moneyness.

<table>
<thead>
<tr>
<th>Product/Model</th>
<th>Without CS</th>
<th>With CS</th>
<th>With CS and SV / $\alpha = 1$</th>
<th>With CS and SV / $\alpha = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread Option -10</td>
<td>12.46%</td>
<td>12.31%</td>
<td>12.45%</td>
<td>12.92%</td>
</tr>
<tr>
<td>Spread Option -5</td>
<td>9.00%</td>
<td>8.85%</td>
<td>8.87%</td>
<td>9.47%</td>
</tr>
<tr>
<td>Spread Option 0</td>
<td>6.15%</td>
<td>6.03%</td>
<td>6.20%</td>
<td>6.72%</td>
</tr>
<tr>
<td>Spread Option 5</td>
<td>3.96%</td>
<td>3.87%</td>
<td>4.01%</td>
<td>4.52%</td>
</tr>
<tr>
<td>Spread Option 10</td>
<td>2.40%</td>
<td>2.34%</td>
<td>2.53%</td>
<td>2.90%</td>
</tr>
</tbody>
</table>

Call on Spread : Long Vovol, Short Correl between vols. Stochastic Vol parameters (for three underlyings):

<table>
<thead>
<tr>
<th>$\kappa_S$</th>
<th>$\kappa_L$</th>
<th>$\sigma_S$</th>
<th>$\sigma_L$</th>
<th>$\rho_S$</th>
<th>$\rho_L$</th>
<th>$\rho_{SL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>400.0%</td>
<td>12.5%</td>
<td>350%</td>
<td>100%</td>
<td>-50%</td>
<td>-50%</td>
<td>50%</td>
</tr>
</tbody>
</table>

Structured Equity Research (HSBC)
Focus on correlation products

Zoom on Cancellable

Product generally considered as a simple product but:

- interest rate risk
- dividend risk
- volatility risk (with vanna and volga changing signs!)
- correlation risk
- pin risk (need for smoothing)

Case of 3Y autocall product:

- 3Y product
- three underlyings (World Basket)
- can be cancelled every year at 100
- short Put 100
- Discrete DI 60

<table>
<thead>
<tr>
<th>Model</th>
<th>LV</th>
<th>LSV ($\alpha = 1$)</th>
<th>LV + CS</th>
<th>LSV + CS, $\alpha = 1$</th>
<th>LSV + CS, $\alpha = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basket Cancellable</td>
<td>93.88%</td>
<td>94.14%</td>
<td>93.61%</td>
<td>94.04%</td>
<td>94.02%</td>
</tr>
<tr>
<td>WO Cancellable</td>
<td>86.15%</td>
<td>85.94%</td>
<td>87.15%</td>
<td>87.44%</td>
<td>87.55%</td>
</tr>
</tbody>
</table>

Two main conclusions:

- Correlation Skew and Stochastic Volatility don’t add (cross effect cannot be neglected)
- Price doesn’t depend on correlation between vols
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Main conclusions

Main issues with correlation in equity:

- Constraints: must remain between -1 and 1, must be part of PSD matrices
- Numerous elements compared to volatility
- Illiquid parameter
- Difficult to integrate new dimensions (overlap between baskets)

Next parameter to gain in complexity, but long evolution.

Currently = essentially studied for improved Macro Risk Management.
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References

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- Dupire : Pricing with a smile
- Langnau : Introduction into Local Correlation Modelling
- Sbai-Jourdain : Coupling Index and stocks
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- Lee et all. : Index Volatility surface via moment-matching techniques
- Piterbarg : Markovian projection for volatility calibration
Appendix 1 : Pathwise equality

True Model = believed to be Index model with Local Vol :
\[ \mathbb{E}_t^Q((l_T - K)^+) = C(t, l_t, K, T) \]

\[
dl_t = l_t \sigma(t, l_t) dW_t \\
d\tilde{l}_t = \sum_i w_i S^i_t \sigma_i(t, S^i_t) dW^i_t
\]

One can write :

\[
\mathbb{E}_0^Q(C(T, l_T)) = C(0, l_0) + \int_0^T \mathbb{E}_0^Q(dC) \\
= C(0, l_0) + \frac{1}{2} \int_0^T \mathbb{E}_0^Q(\frac{\partial^2 C}{\partial x^2}(d < \tilde{l}_t, \tilde{l}_t > - d < l_t, l_t >))
\]

Pathwise equality :
\[ d < \tilde{l}_t, \tilde{l}_t >= d < l_t, l_t >. \]
Or :
\[ \sum_{i,j} w_i w_j S^i_t S^j_t \sigma_i(t, S^i_t) \sigma_j(t, S^j_t) \rho_{i,j} = \sigma^2(t, l_t). \]
Sufficient condition, but not necessary.
Other sufficient condition (but still not necessary for models like in Lucic 2009) :
\[
\mathbb{E}_0^Q(\sum_{i,j} w_i w_j S^i_t S^j_t \sigma_i(t, S^i_t) \sigma_j(t, S^j_t) \rho_{i,j}^{Loc}(S^1_t, \ldots, S^n_t) | S^1_t, \ldots, S^n_t) = \mathbb{E}_0^Q(\sum_{i,j} w_i w_j \sqrt{V^i_t} \sqrt{V^j_t} \rho_{i,j}^{Sto} | S^1_t, \ldots, S^n_t)
\]
Appendix 2 : WO formula demonstration(1)

Ito-Tanaka to WO Call (abusive notations) :

\[
f(x_1, \ldots, x_n) = \left( \sum_{l} x_l \prod_{k \neq l} 1_{x_l < x_k} - K \right)^+
\]

\[
\frac{\partial f}{\partial x_i} = \prod_{k \neq i} 1_{x_i < x_k} 1_{x_i \geq K} (\text{cf. other terms cancel out})
\]

\[
\frac{\partial^2 f}{\partial x_i^2} = \prod_{k \neq i} 1_{x_i < x_k} 1_{x_i = K} - \left( \sum_{j \neq i} \left( \prod_{k \neq i, k \neq j} 1_{x_i < x_k} 1_{x_i = x_j} \right) 1_{x_i \geq K} \right)
\]

\[
\frac{\partial^2 f}{\partial x_i \partial x_j} = \prod_{k \neq i, k \neq j} 1_{x_i < x_k} 1_{x_i = x_j} 1_{x_i \geq K}
\]
Appendix 2: WO formula demonstration(2)

Deterministic interest rates:

\[
d\mathbb{E}^Q \left( \exp\left(- \int_0^t r_s ds\right) f(S^1_t, \ldots, S^1_t) \right) = \frac{\partial C}{\partial t} dt \\
= -r_tC dt + \sum_i \mathbb{E}^Q \left( \exp\left(- \int_0^t r_s ds\right) \frac{\partial f}{\partial x_i} (r_t - q^i_t) S^i_t \right) dt \\
+ \frac{1}{2} \sum_i \mathbb{E}^Q \left( \exp\left(- \int_0^t r_s ds\right) \frac{\partial^2 f}{\partial x_i^2} d < S^i_t > \right) \\
+ \sum_{j>i} \mathbb{E}^Q \left( \exp\left(- \int_0^t r_s ds\right) \frac{\partial^2 f}{\partial x_i \partial x_j} d < S^i_t, S^j_t > \right)
\]

Rearranging terms give final result.
Appendix 2: WO formula demonstration (3)

Same formula for BO options:

$$\frac{\partial C_{BO}}{\partial t} = -r_tC_{BO} + (r_t - q_t^{BO}(K))(C_{BO} - K) + \frac{1}{2}K^2 \frac{\partial^2 C_{BO}}{\partial K^2} \mathbb{E}^Q(\sigma_{BO}^2(t,K) | BO = K) + \frac{1}{2} \sum_{i > j} \mathbb{E}^Q \left( \frac{d < S^i_t - S^j_t, S^i_t - S^j_t >}{dt} \delta_{S^i_t = S^j_t} \mathbb{1}_{BO_t > K} \mathbb{1}_{BO_t = S^i_t} \right)$$

and Rainbow (calculation is awful):

$$Rbw(S^1, \ldots, S^n) = \sum_j w_j S(j) \text{ with } S^{(1)} \geq \ldots \geq S^{(n)}$$

$$\frac{\partial C_{Rbw}}{\partial t} = -r_tC_{Rbw} + r_t(C_{Rbw} - K) \frac{\partial C_{Rbw}}{\partial K} - \mathbb{E}^Q(\sum_i \sum_j w_j q_t^i S(j) \mathbb{1}_{S(j) = S^i_{Rbw} > K})$$

$$+ \frac{1}{2} \frac{\partial^2 C_{Rbw}}{\partial K^2} \mathbb{E}^Q \left( \sum_{i,j} w_k w_l S(k) = S^i, S(l) = S^j \frac{d < S^i_t, S^j_t >}{dt} \bigg| \tilde{Rbw} = K \right)$$

$$- \frac{1}{2} \sum_{i > j} \mathbb{E}^Q \left( \frac{d < S^i_t - S^j_t, S^i_t - S^j_t >}{dt} \delta_{S^i_t = S^j_t} \mathbb{1}_{Rbw > K} \sum_l (w_{l+1} - w_l) \mathbb{1}_{S(l) = S^i_t} \right)$$

Last term disappears and other terms equal to basket equation for equally weighted baskets.
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